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NOTES ON MATRIX THEORY-IX

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SUDMARY

Using a generalization of an identity of Siegel, a concavity theorem is established for power products of the form $|\mathbf{X}_1|^{\mathbf{a}_1} |\mathbf{X}_2|^{\mathbf{a}_2} \cdots |\mathbf{X}_n|^{\mathbf{a}_n}$, where $|\mathbf{X}_k| - |\mathbf{x}_{ij}|$, 1, j = 1, 2, ..., k.

NOTES ON MATRIX THEORY---IX

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§1. Introduction

In a recent note, [1], we showed that the inequality

$$(1) \qquad |\lambda A + (1-\lambda)B| \geq |A|^{\lambda} |B|^{1-\lambda},$$

valid for positive definite matrices A and B, for $0 \le \lambda \le 1$, was a simple consequence of Holder's inequality and the identity

(2)
$$\int_{0}^{\infty} e^{-(x,cx)} \prod_{1} dx_{1} = \sqrt{\tau}^{n}/|c|^{1/2}$$
,

for C a positive definite matrix of order n.

In this note we wish to use a more recondite identity, a generalization of an integral of Ingham and Siegel, due to A. Selberg, to derive an extensive generalization of (1), namely

Theorem. Let A and B be two positive definite matrices of order n, and let $C = \lambda A + (1 - \lambda)B$, for $0 \le \lambda \le 1$. For each j = 1, 2, ..., n, let $A^{(j)}$ denote the principal submatrix of A obtained by deleting the first (j-1) rows and columns, $(in \ particular, A^{(1)} = A)$. Let $B^{(j)}$, $C^{(j)}$, have similar meanings. If $k_1, k_2, ..., k_n$ are n real numbers such that

(3)
$$\sum_{i=1}^{J} k_i \geq 0$$
, $j = 1, 2, ..., n$

then

The above sharp form of the inequality is due to a referee. We shall first present his proof below, and then the proof of a particular case, derived from the identity mentioned above.

§2. Proof of Theorem

According to Bergstrom's inequality, [2], or a minimum theorem due to Fan [4], we have for j = 1, 2, ..., n-1,

(1)
$$\frac{|C^{(J)}|}{|C^{(J+1)}|} \ge \frac{|\lambda A^{(J)}|}{|\lambda A^{(J+1)}|} + \frac{|(1-\lambda)B^{(J)}|}{|(1-\lambda)B^{(J+1)}|}$$
$$= \lambda \frac{|A^{(J)}|}{|A^{(J+1)}|} + (1-\lambda) \frac{|B^{(J)}|}{|B^{(J+1)}|}$$
$$\ge \left(\frac{|A^{(J)}|}{|A^{(J+1)}|}\right)^{\lambda} \left(\frac{|B^{(J)}|}{|B^{(J+1)}|}\right)^{(1-\lambda)}.$$

The desired inequality follows upon writing

$$\cdots \left(\frac{|c^{(n-1)}|}{|c^{(n)}|}\right)^{k_1+k_2+\cdots+k_{n-1}}|c^{(n)}|^{\sum_{i=1}^{n}k_i}.$$

and using the condition that $\sum_{i=1}^{k_i} k_i \ge 0$, together with the inequality above.

3. Partial Proof

It was shown by Siegel, [6], p. 44, that the following generalization of the gamma function integral exists:

(1)
$$\int_{x>0} e^{-tr(xy)} |x| = \frac{(n-1)}{1} dx_{1,1} = e_n |y|^{-\epsilon}.$$

Here X and Y are symmetric matrices of order n, with Y positive definite, and the integration is extended over the region of $x_{i,j}$ —space in which X is positive definite.

The constant an is given by

(2)
$$a_n = r \frac{n(n-1)}{4} P(s) P(s - \frac{1}{2}) \cdots P(s - (\frac{n-1}{2})).$$

The integral converges for $Re(s) > (\frac{n-1}{2})$, and equals the right-hand side.

It was pointed out to the author by A. Selberg that an extension of Siegel's integral exists, namely

(3)
$$\int_{\mathbf{x}}^{n} e^{-\mathbf{cr}(\mathbf{x}\mathbf{y})|\mathbf{x}^{(1)}|^{\frac{1}{n-1}} |\mathbf{x}_{1} - (\frac{n+1}{2})} |\mathbf{x}^{(2)}|^{-k_{1}} \cdots |\mathbf{x}^{(n)}|^{-k_{n-1}} |\mathbf{d}\mathbf{x}_{1}|^{\frac{1}{n-1}}$$

$$= b_{n} |\mathbf{x}_{n}|^{-k_{n}} |\mathbf{x}_{n-1}|^{-k_{n-1}} \cdots |\mathbf{x}_{1}|^{-k_{1}},$$

where $X^{(j)}$ is as above, $Y_j = (y_{ij}), 1 \le i, j \le k$, and

(4)
$$b_n = \pi \frac{n(n-1)}{4} P(k_n) P(k_n + k_{n-1} - \frac{1}{2}) \cdots P(\frac{n}{n-1} k_1 - (\frac{n+1}{2}))$$
.

The integral exists and has the stated value provided that each of the expressions k_n , $k_n + k_{n-1} - \frac{1}{2}$, ... $\sum_{i=1}^{n} k_i - (n + \frac{1}{2})$ is positive. Once we have a representation for $|Y_n|^{-k_1} - |Y_{n-1}|^{-k_1} = \psi(Y)$ in the form

(5)
$$\forall (Y) = \int_{X>0} \phi(X)e^{-tr(XY)} \prod_{1 \leq j} dx_{1j},$$

with $\phi \geq 0$, the proof proceeds as in [1].

A proof of (3) and an analogous extension of an integral of Ingham [5] equivalent to Siegel's may be found in [3], together with some applications.

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